A Dempster-Shafer Bayesian Solution to the Banff A1 Challenge

Paul Edlefsen

August 1, 2007

The Three Poisson Model

$$n \sim \mathcal{P}\mathrm{ois}(\epsilon\, s + b\,)$$

$$y \sim \mathcal{P}\mathrm{ois}(t\, b\,)$$

$$z \sim \mathcal{P}\mathrm{ois}(u\, \epsilon\,)$$

Overview

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- The Plausibility Transform

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- The Plausibility Transform
- An Analytical Form of the Solution

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b and ϵ are nuisance parameters.

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$$n_i \sim \mathcal{P}\text{ois}(\epsilon_i s + b_i)$$

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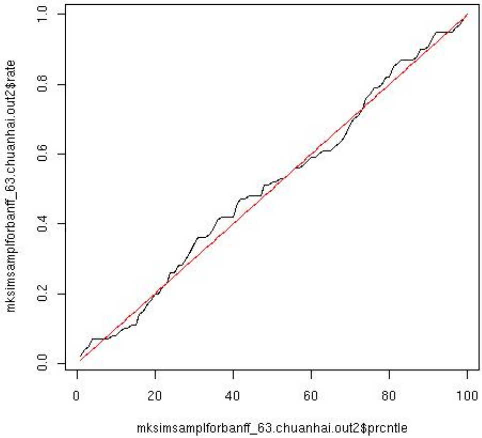
i is the channel, in 1..n.

Some Results



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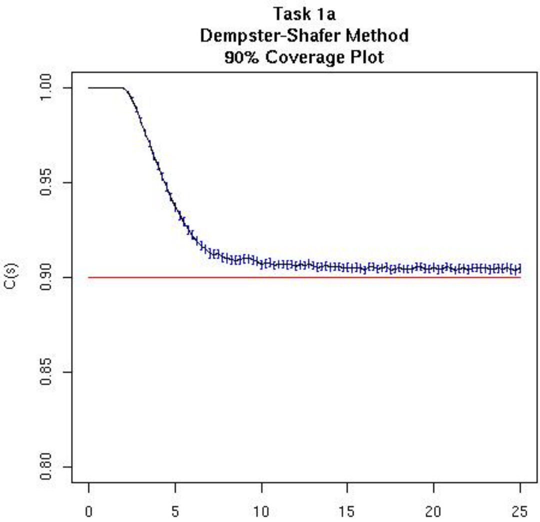

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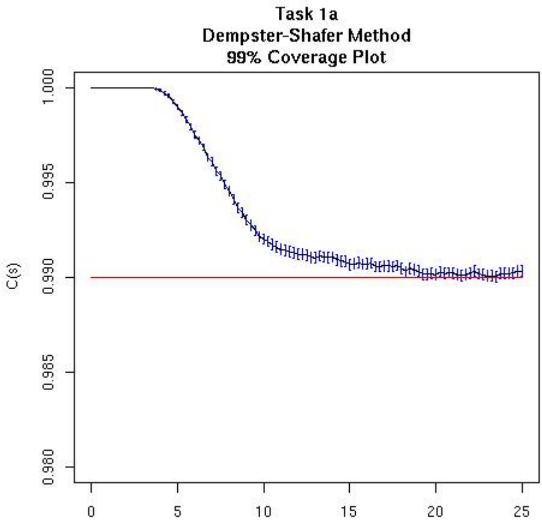

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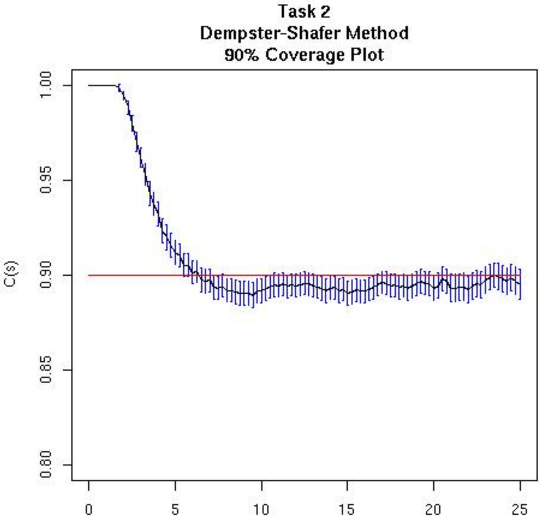

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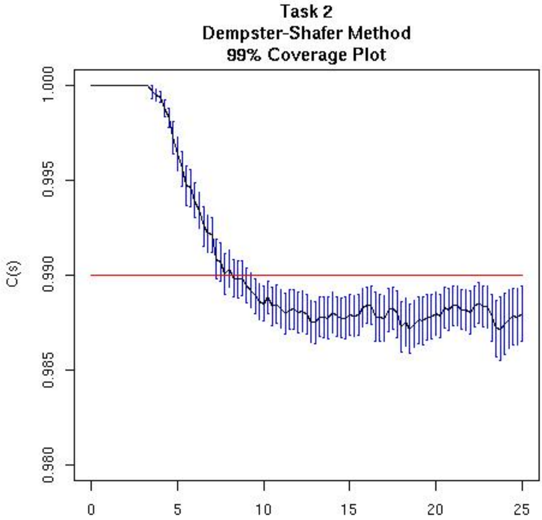

The Poisson DSM

$$K \sim \mathcal{P}\text{ois}(\lambda)$$

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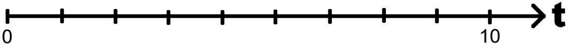

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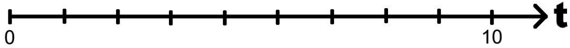

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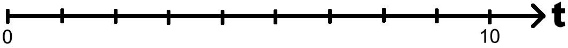
$$K \sim \mathcal{P}\text{ois}(\lambda_{\mathbf{t}})$$



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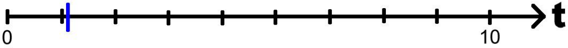

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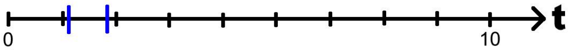

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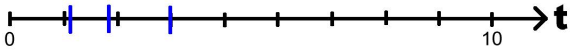
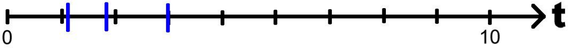


The Poisson DSM

$$K \sim \mathcal{P}\text{ois}(t)$$



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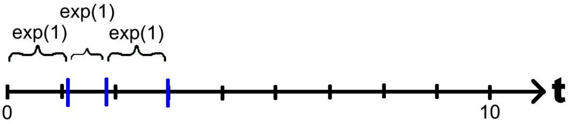
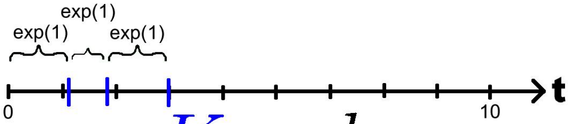


The Poisson DSM

$$K \sim \mathcal{P}\text{ois}(\lambda)$$



$$K = k$$

The Poisson DSM

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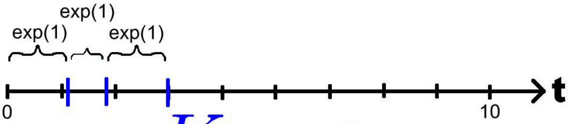
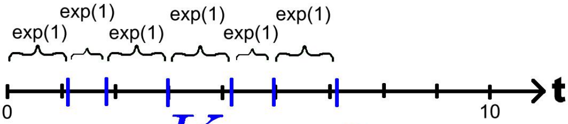


$$K = 6$$

The Poisson DSM

$$K \sim \mathcal{P}\text{ois}(\lambda)$$



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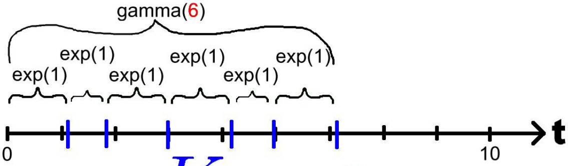

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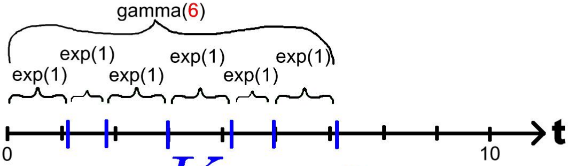
$$K \sim \mathcal{P}\text{ois}(\lambda)$$

$$\Lambda_l \leq \lambda \leq \Lambda_u$$

$$\Lambda_l \sim \mathcal{G}\text{amma}(6)$$



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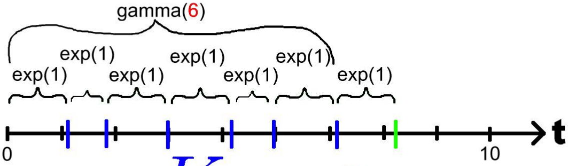

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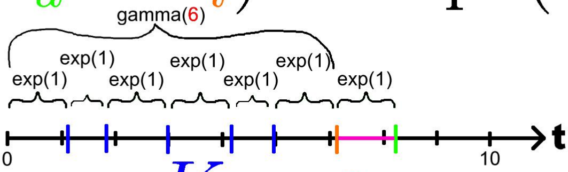

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The Three Poisson Model

The parameter of interest is s.

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b and ϵ are nuisance parameters.

i is the channel, in 1..n.

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$$\mathbf{Z}_l^i \leq u_i \epsilon_i \leq \mathbf{Z}_u^i$$

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$$n_i \sim \mathcal{P}\mathrm{ois}(\epsilon_i s + b_i)$$

$$\mathbf{Y}_l^i \leq t_i b_i \leq \mathbf{Y}_u^i$$

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$$\mathbf{N}_l^i \leq (\epsilon_i s + b_i) \leq \mathbf{N}_u^i$$

$$\mathbf{N}_l^i - \frac{1}{t_i}\mathbf{Y}_u^i \leq \epsilon_i s \leq \mathbf{N}_u^i - \frac{1}{t_i}\mathbf{Y}_l^i$$

$$\frac{1}{u_i}\mathbf{Z}_l^i \leq \epsilon_i \leq \frac{1}{u_i}\mathbf{Z}_u^i$$

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$$s \geq 0$$

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$$s \geq 0$$

$$s_l \leq s \leq \frac{\mathbf{N}_u^i - \frac{1}{t_i}\mathbf{Y}_l^i}{\frac{1}{u_i}\mathbf{Z}_l^i}$$

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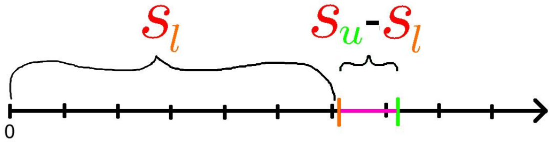
$$s \geq 0$$

$$S_l \leq s \leq S_u$$

The Plausibility Transform

$$S_l \leq s \leq S_u$$

$$s \geq 0$$



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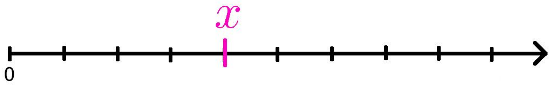
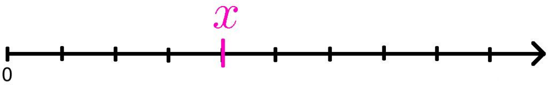


The Plausibility Transform

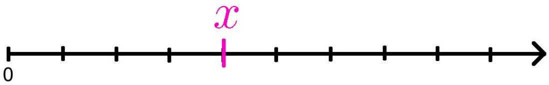
$$\text{Plaus}(\{x\}) = \mathbb{P}(x \in (\mathbf{S}_l, \mathbf{S}_u))$$



THE PLAUSIBILITY TRANSFORM

$$\text{Plaus}(\{x\}) = \mathbb{P}(x \in (\mathbf{S}_l, \mathbf{S}_u))$$

$$= 1 - \mathbb{P}(x \notin (\mathbf{S}_l, \mathbf{S}_u))$$



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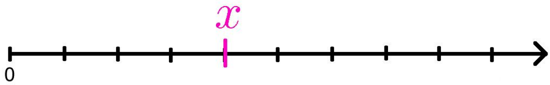
$$= 1 - (F_{\mathbf{S}_u}(x) + (1 - F_{\mathbf{S}_l}(x)))$$



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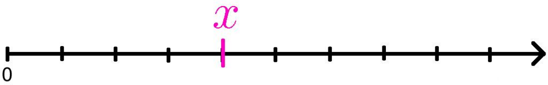
$$= F_{\mathbf{S}_l}(x) - F_{\mathbf{S}_u}(x)$$



Analytical Form

From these and from the additional constraint that $s \geq 0$, we see that

$$\mathbf{S}_l^i = \frac{\max(0, \mathbf{N}_l^i - \frac{1}{t}\mathbf{Y}_u^i)}{\frac{1}{u}\mathbf{Z}_u^i} \text{ and}$$

$$\mathbf{S}_u^i = \frac{\mathbf{N}_u^i - \frac{1}{t}\mathbf{Y}_l^i}{\frac{1}{u}\mathbf{Z}_l^i}$$

in the equation $\mathbf{S}_l^i \leq s \leq \mathbf{S}_u^i$.

Analytical Form

Thus, if we ignore (momentarily) the constraint that $s \geq 0$, we may characterize the CDFs of $\mathbf{S}_l^i$ and $\mathbf{S}_u^i$ as

$$F^*_{\mathbf{S}_l^i}(x) = \mathbb{P}\left(\frac{\mathbf{N}_l^i - \frac{1}{t_i}\mathbf{Y}_u^i}{\frac{1}{u_i}\mathbf{Z}_u^i} \leq x\right) \text{ and}$$

$$F^*_{\mathbf{S}_u^i}(x) = \mathbb{P}\left(\frac{\mathbf{N}_u^i - \frac{1}{t_i}\mathbf{Y}_l^i}{\frac{1}{u_i}\mathbf{Z}_l^i} \leq x\right).$$

Analytical Form

Rearranging, we may write this as

$$F^*_{\mathbf{S}^i_l}(x) = \mathbb{P}(\mathbf{N}^i_l \leq \frac{1}{t_i}\mathbf{Y}^i_u + \frac{x}{u_i}\mathbf{Z}^i_u) \text{ and}$$

$$F^*_{\mathbf{S}^i_u}(x) = \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l).$$

Unnormalized CDFs

We will see that these can be written in terms of the Beta distribution as

$$F^*_{\mathbf{S}^i_u}(x) = 1 - \text{pBeta}(\frac{u_i}{u_i + x}, z_i, n_i + 1) +$$

$$\int_0^{\frac{u_i}{u_i+x}} \text{pBeta}(\frac{1}{1+t_i - \frac{\gamma}{\frac{u_i}{u_i+x}}}, z_i + n_i + 1, y_i)\text{dBeta}(\gamma, z_i, n_i + 1)d\gamma,$$

and

$$F^*_{\mathbf{S}^i_l}(x) = 1 - \text{pBeta}(\frac{u_i}{u_i + x}, z_i + 1, n_i) +$$

$$\int_0^{\frac{u_i}{u_i+x}} \text{pBeta}(\frac{1}{1+t_i - \frac{\gamma}{\frac{u_i}{u_i+x}}}, z_i + 1 + n_i, y_i + 1)\text{dBeta}(\gamma, z_i + 1, n_i)d\gamma.$$

Analytical Form

We are ultimately interested in the normalized quantities

$$\mathsf{F}_{\mathbf{S}_l^i}(x) = \frac{\mathsf{F}^*_{\mathbf{S}_l^i} - \mathbb{P}(\mathbf{S}_u^i < 0)}{1 - \mathbb{P}(\mathbf{S}_u^i < 0)} \text{ and}$$

$$\mathsf{F}_{\mathbf{S}_u^i}(x) = \frac{\mathsf{F}^*_{\mathbf{S}_u^i} - \mathbb{P}(\mathbf{S}_u^i < 0)}{1 - \mathbb{P}(\mathbf{S}_u^i < 0)},$$

where we condition on the upper end of the interval, $\mathbf{S}_u^i$, being non-negative.

Analytical Form

Since this condition is met whenever $\mathbf{N}_u^i \geq \frac{1}{t_i}\mathbf{Y}_l^i$, we have

$$F_{\mathbf{S}_l^i}(x) = \frac{\mathbb{P}(\mathbf{N}_l^i \leq \frac{1}{t_i}\mathbf{Y}_u^i + \frac{x}{u_i}\mathbf{Z}_u^i) - \mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i)}{1 - \mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i)} \text{ and}$$
$$F_{\mathbf{S}_u^i}(x) = \frac{\mathbb{P}(\mathbf{N}_u^i \leq \frac{1}{t_i}\mathbf{Y}_l^i + \frac{x}{u_i}\mathbf{Z}_l^i) - \mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i)}{1 - \mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i)}. \quad (1)$$

Conflict

The constraint $s \geq 0$ is violated whenever $\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i$, so the probability of conflict is $\mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i)$.

Conflict

Since $\mathbf{N}_u^i$ and $\mathbf{Y}_l^i$ are gamma-distributed with unit scale, and with shape parameters $(n_i + 1)$ and y_i, respectively,

$$\frac{\mathbf{Y}_l^i}{\mathbf{Y}_l^i + \mathbf{N}_u^i} \sim \mathcal{B}\text{eta}(y_i, n_i + 1).$$

Conflict

We can thus rearrange the probability of conflict to utilize the CDF of a beta:

$$\begin{aligned}
\mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i) &= \mathbb{P}(\frac{\mathbf{N}_u^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i} < \frac{\frac{1}{t_i}\mathbf{Y}_l^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i}) \\
&= \mathbb{P}(1 - \frac{\mathbf{Y}_l^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i} < \frac{\frac{1}{t_i}\mathbf{Y}_l^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i}) \\
&= \mathbb{P}(\frac{(\frac{1}{t_i} + 1)\mathbf{Y}_l^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i} > 1) \\
&= \mathbb{P}(\frac{\mathbf{Y}_l^i}{\mathbf{N}_u^i + \mathbf{Y}_l^i} > \frac{b}{b+1}).
\end{aligned}$$

Conflict

So the probability of conflict is

$$\mathbb{P}(\mathbf{N}_u^i < \frac{1}{t_i}\mathbf{Y}_l^i) = 1 - \text{pBeta}(\frac{t_i}{t_i + 1}, y_i, n_i + 1), \tag{2}$$

where pBeta$(\cdot, \alpha, \beta)$ is the CDF of a beta with parameters α and β.

Unnormalized CDFs

We now use similar techniques to characterize the unnormalized components $\mathsf{F}^*_{\mathbf{S}^i_u}(\cdot)$ and $\mathsf{F}^*_{\mathbf{S}^i_u}(\cdot)$. Consider first

$$\mathsf{F}^*_{\mathbf{S}^i_u}(x) = \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l).$$

Unnormalized CDFs

Noting that gamma random variables are never negative, we may apply the law of total probability to rewrite this as

$$F^*_{\mathbf{S}^i_u}(x) = \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l \text{ and } \mathbf{N}^i_u \leq \frac{x}{u_i}\mathbf{Z}^i_l) + \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l$$

$$= \mathbb{P}(\mathbf{N}^i_u \leq \frac{x}{u_i}\mathbf{Z}^i_l) + \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l \text{ and } \mathbf{N}^i_u > \frac{x}{u_i}\mathbf{Z}^i_l)$$

$$= \mathbb{P}(\frac{\mathbf{Z}^i_l}{\mathbf{Z}^i_l + \mathbf{N}^i_u} > \frac{u_i}{u_i + x}) + \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l \text{ and } \mathbf{N}^i_u > \frac{x}{u_i}\mathbf{Z}^i_l$$

$$= 1 - \text{pBeta}(\frac{u_i}{u_i + x}, z_i, n_i + 1) + \mathbb{P}(\mathbf{N}^i_u \leq \frac{1}{t_i}\mathbf{Y}^i_l + \frac{x}{u_i}\mathbf{Z}^i_l \text{ and } \mathbf{N}^i_u >$$

by an argument similar to that used in deriving the Beta CDF representation for the probability of conflict.

Unnormalized CDFs

The latter part may be simplified also. We can rewrite $\mathbf{N}_u^i - \frac{x}{u_i}\mathbf{Z}_l^i$ as $(1 - \frac{\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i}}{\frac{u_i}{u_i+x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i)$, since

$$
\begin{aligned}
(1 - \frac{\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i}}{\frac{u_i}{u_i+x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i) &= (\mathbf{Z}_l^i + \mathbf{N}_u^i) - \frac{\mathbf{Z}_l^i}{\frac{u_i}{u_i+x}} \\
&= (\mathbf{Z}_l^i + \mathbf{N}_u^i) - \frac{\mathbf{Z}_l^i}{\frac{1}{1+\frac{x}{u_i}}} \\
&= (\mathbf{Z}_l^i + \mathbf{N}_u^i) - \mathbf{Z}_l^i(1 + \frac{x}{u_i}) \\
&= \mathbf{N}_u^i - \frac{x}{u_i}\mathbf{Z}_l^i.
\end{aligned}
$$

Unnormalized CDFs

This leads to

$$\mathbb{P}(\mathbf{N}_u^i \leq \frac{1}{t_i}\mathbf{Y}_l^i + \frac{x}{u_i}\mathbf{Z}_l^i \text{ and } \mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i)$$

$$= \mathbb{P}\left(\frac{1}{t_i}\mathbf{Y}_l^i \geq (1 - \frac{\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i + \mathbf{N}_u^i}}{\frac{u_i}{u_i + x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i) \text{ and } \mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i\right).(3)$$

Unnormalized CDFs

Recognizing again that the event that $\mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i$ is the same as the event that $\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i} \le \frac{u_i}{u_i+x}$, the complicated probabilty in (3) may be simplified by conditioning on the value of $\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i} = \gamma$:

$$\mathbb{P}(\mathbf{N}_u^i \le \frac{1}{t_i}\mathbf{Y}_l^i + \frac{x}{u_i}\mathbf{Z}_l^i \text{ and } \mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i)$$

$$= \mathbb{P}(\frac{1}{t_i}\mathbf{Y}_l^i \ge (1 - \frac{\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i}}{\frac{u_i}{u_i+x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i) \text{ and } \mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i)$$

$$= \mathbb{P}(\frac{1}{t_i}\mathbf{Y}_l^i \ge (1 - \frac{\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i+\mathbf{N}_u^i}}{\frac{u_i}{u_i+x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i) \text{ and } \frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i + \mathbf{N}_u^i} \le \frac{u_i}{u_i + x})$$

$$= \int_0^{\frac{u_i}{u_i+x}} \mathbb{P}(\frac{1}{t_i}\mathbf{Y}_l^i \ge (1 - \frac{\gamma}{\frac{u_i}{u_i+x}})(\mathbf{Z}_l^i + \mathbf{N}_u^i) \Big| \frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i + \mathbf{N}_u^i} = \gamma) d\mathbb{P}(\frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i + \mathbf{N}_u^i} <$$

Unnormalized CDFs

and since $(\mathbf{Z}_l^i + \mathbf{N}_u^i) \perp\!\!\!\perp \frac{\mathbf{Z}_l^i}{\mathbf{Z}_l^i + \mathbf{N}_u^i}$, we get

$$\mathbb{P}(\mathbf{N}_u^i \leq \frac{1}{t_i}\mathbf{Y}_l^i + \frac{x}{u_i}\mathbf{Z}_l^i \text{ and } \mathbf{N}_u^i > \frac{x}{u_i}\mathbf{Z}_l^i)$$

$$= \int_0^{\frac{u_i}{u_i+x}} \text{pBeta}(\frac{1}{1+t_i-\frac{\gamma}{\frac{u_i}{u_i+x}}}, z_i + n_i + 1, y_i)\text{dBeta}(\gamma, z_i, n_i + 1)d\gamma,$$

where $\text{dBeta}(\cdot, \alpha, \beta)$ is the pdf of a beta with parameters α and β. Note that this can be approximated to any desired precision by a straightforward numerical integration.

Unnormalized CDFs

Thus we have

$$F^*_{\mathbf{S}^i_u}(x) = 1 - \text{pBeta}(\frac{u_i}{u_i + x}, z_i, n_i + 1) +$$

$$\int_0^{\frac{u_i}{u_i+x}} \text{pBeta}(\frac{1}{1+t_i-\frac{\gamma}{\frac{u_i}{u_i+x}}}, z_i + n_i + 1, y_i)\text{dBeta}(\gamma, z_i, n_i + 1)d\gamma, (4)$$

and, by an analogous derivation,

$$F^*_{\mathbf{S}^i_l}(x) = 1 - \text{pBeta}(\frac{u_i}{u_i + x}, z_i + 1, n_i) +$$

$$\int_0^{\frac{u_i}{u_i+x}} \text{pBeta}(\frac{1}{1+t_i-\frac{\gamma}{\frac{u_i}{u_i+x}}}, z_i + 1 + n_i, y_i + 1)\text{dBeta}(\gamma, z_i + 1, n_i)d\gamma. (5)$$

Normalized CDFs

Plugging equations (4), (5), and (2) into equation (1) concludes the derivation, the CDFs of the upper and lower bounds on $\mathbf{S}$ induced by the evidence from a single channel. Combining evidence across channels and normalizing yields the D-S Bayesian posterior

$$f_{\mathbf{S}}(x) = \frac{\prod_{i=1}^{n} \left(F_{\mathbf{S}_l^i}(x) - F_{\mathbf{S}_u^i}(x) \right)}{\int_{x=0}^{\infty} \prod_{i=1}^{n} \left(F_{\mathbf{S}_l^i}(x) - F_{\mathbf{S}_u^i}(x) \right)}. \tag{6}$$

Special Cases

The above derivation for $r_i(\cdot)$ assumed that n_i, y_i, and z_i are all positive. In the event that $n_i = 0$, $\mathbf{S}_l^i = 0$ and $\mathsf{F}_{\mathbf{S}_l^i}(s) = 1 \; \forall x \geq 0$.
When $y_i = 0$, there is no conflict, and
$\mathsf{F}^*_{\mathbf{S}_u^i}(x) = 1 - \text{pBeta}(\frac{u_i}{u_i+x}, z_i, n_i + 1)$, unless $z_i = 0$. Whenever $z_i = 0$, $\mathbf{S}_u^i = \infty$, so $\mathsf{F}_{\mathbf{S}_u^i}(x) = 0 \; \forall x < \infty$.
Note that when $n_i = z_i = 0$, we learn nothing new about s.