Power-Law Estimation via MPS

Theory and Applications

Joshua D. Ingram[1, 2] Vinay Kashyap [1] Xufei Wang[3]
Xiao-Li Meng[3] Bernhard Klingenberg[2]

[1]High Energy Astrophysics Division
Center for Astrophysics | Harvard & Smithsonian

[2]Department of Data Science
New College of Florida

[3]Department of Statistics
Harvard University

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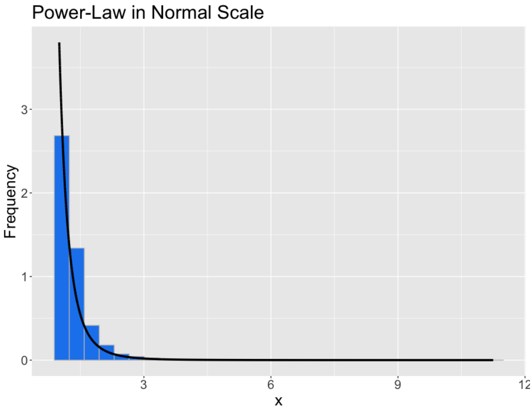
Power-Law Distributions in Solar Flares

Solar flare properties have been found to follow power-law distributions (Lu & Hamilton 1991, Wheatland & Litvinenko 2002, Clauset et al. 2009). These properties include total energy, peak flux, waiting times, etc.

- The functional form of a power-law is given by $f(x) = cx^{-\alpha}$.



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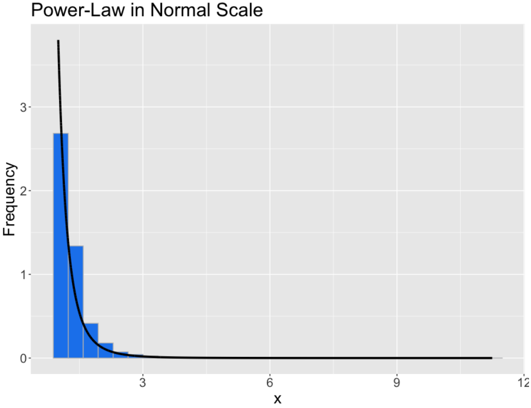
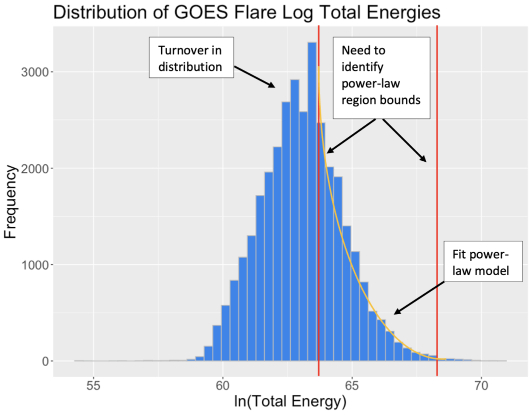
- In log scale, where $y = \log(x)$, it has the form $g(y) = ke^{-y(\alpha-1)}$.



Turnover in the Distribution

- A power-law is only observed over a bounded range to the right of the mode.



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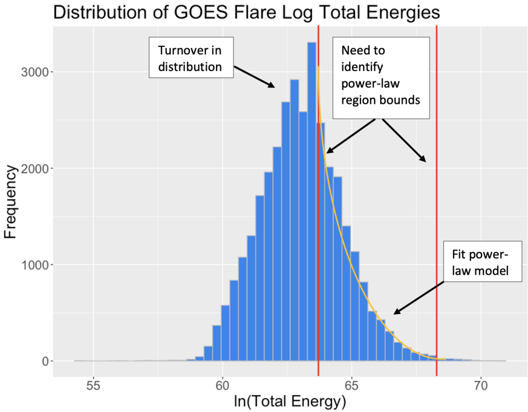
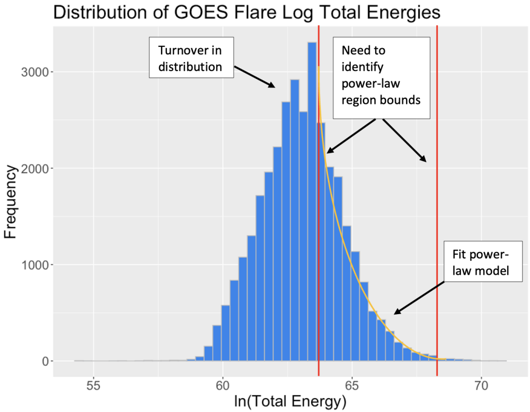


Turnover in the Distribution

- A power-law is only observed over a bounded range to the right of the mode.
- The turnover observed at lower values is the result of missing data caused by detection sensitivity.
- The start of the power-law is typically visually determined, or by using the K-S statistic (see Clauset et al. 2009).



The Problem

How do we accurately determine the power-law region and estimate α?

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Maximum Product of Spacings Method

- For $X_1 < X_2 < \ldots < X_n$, fit a CDF that maximizes the product of spacings: $\prod_{i=0}^{n}\{F(X_{i+1}) - F(X_i)\}$.

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- We fit a parametric MPS model, and add a penalty term.

Maximum Product of Spacings Method

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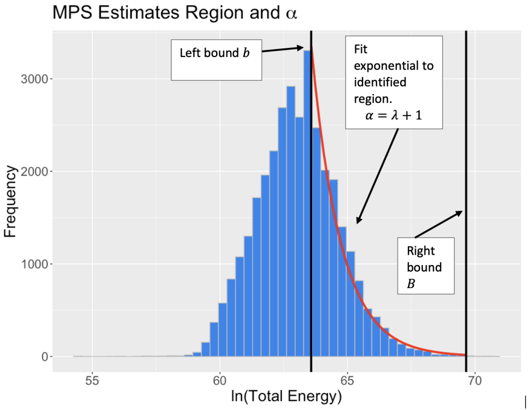

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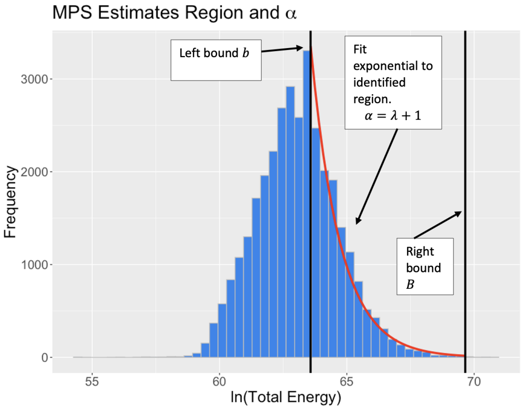
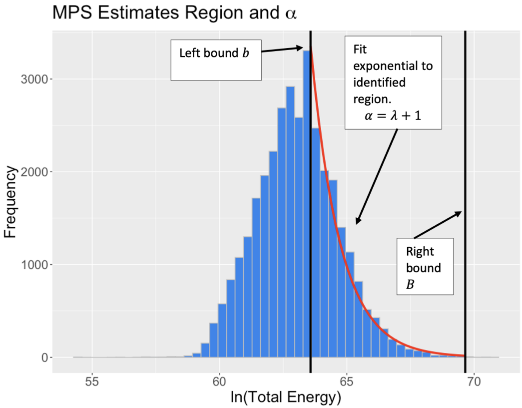
- Within the region b and B, the data follows an exponential distribution with rate λ.



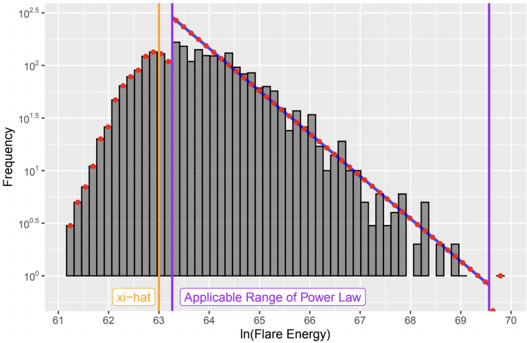
Maximum Product of Spacings Method

- Take the logarithm of the total energy. Power-law region becomes exponential.
- Within the region b and B, the data follows an exponential distribution with rate λ.
- The Maximum Product of Spacings method estimates these bounds and the rate parameter λ. $(\alpha = \lambda + 1)$



Alternative: Hybrid Parametric Multinomial Model

- Given k pre-specified energy bins, fit a multinomial model using bin probabilities.



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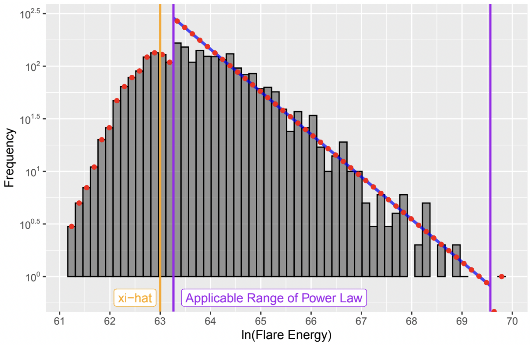

Alternative: Hybrid Parametric Multinomial Model

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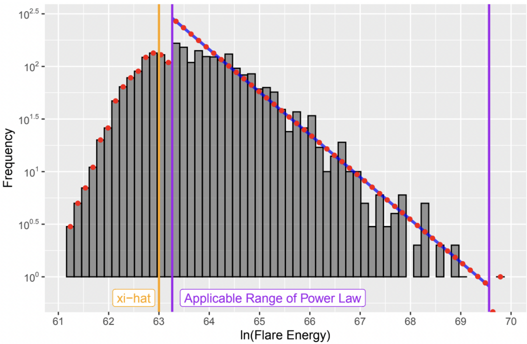
- Bin probabilities within the power-law region are parameterized by the power-law, outside are not.
- Maximize the likelihood function, estimate α and lower bound ξ.



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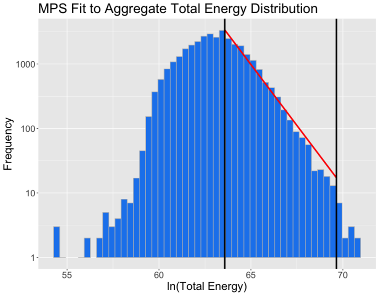
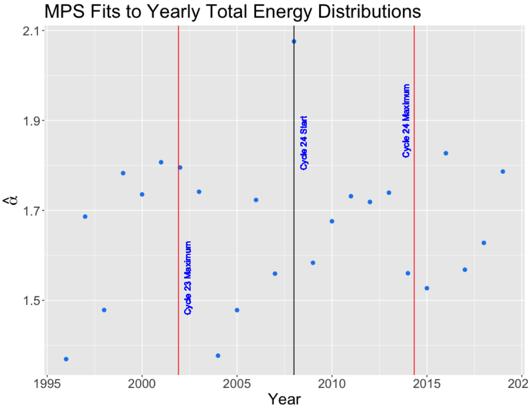
Total Energy - Aggregate and By-Cycle

Data	$\hat{\alpha}$	$\hat{b}$	$\hat{B}$	n
Aggregate	1.87	63.58	69.65	33445
Cycle 23	1.85	63.58	69.09	18457
Cycle 24	1.64	62.61	67.10	14988



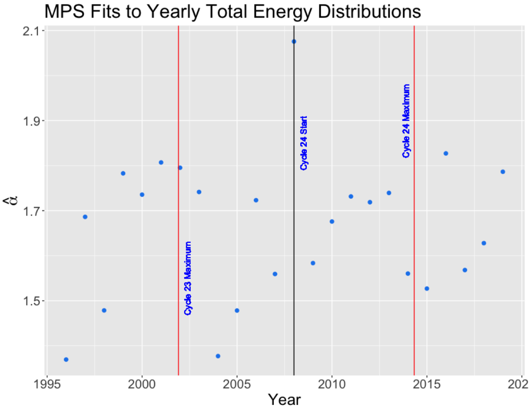
Total Energy - Yearly

- $\hat{\alpha}$ varies between 1.37 and 1.87 throughout the two solar cycles (outlier of $\hat{\alpha} = 2.07$ in 2008, where $n = 88$)



Total Energy - Yearly

- $\hat{\alpha}$ varies between 1.37 and 1.87 throughout the two solar cycles (outlier of $\hat{\alpha} = 2.07$ in 2008, where $n = 88$)
- No significant trend in $\hat{\alpha}$ within a solar cycle.



Summary

- We have further developed our model based on the MPS method to estimate the lower and upper bounds of the power-law region, and α.

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- For aggregate data, $\hat{\alpha} = 1.87$. Similar to previous estimates close to 1.8.
- For yearly data, $\hat{\alpha}$ varies between 1.37 and 1.87. We identify no clear trend.

Moving Forward

- Further study performance of MPS through simulations.

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- Fit our power-law models to solar flare property distributions (total energy, peak flux, waiting time).

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- Further study performance of MPS through simulations.
- Fit our power-law models to solar flare property distributions (total energy, peak flux, waiting time).
- Compare power-law models and verify results.

References

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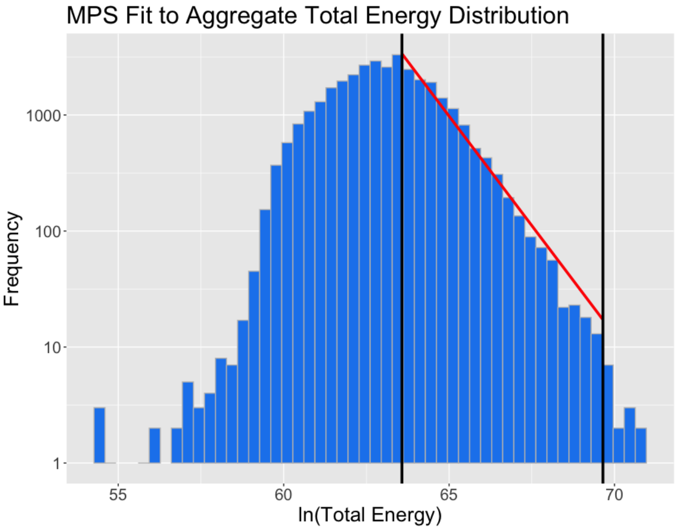
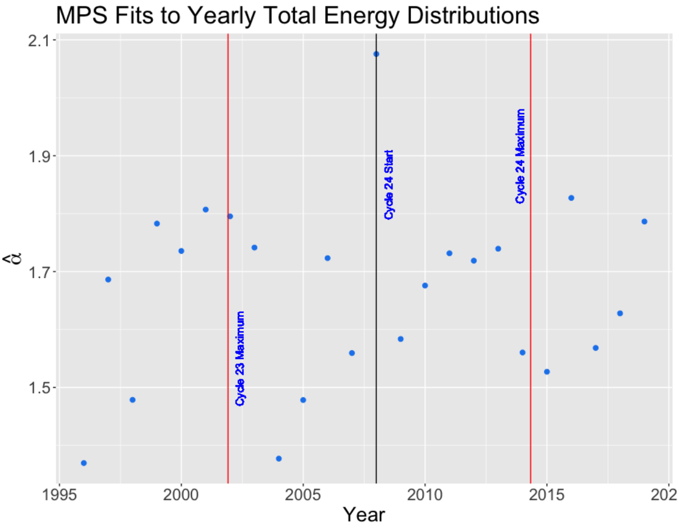
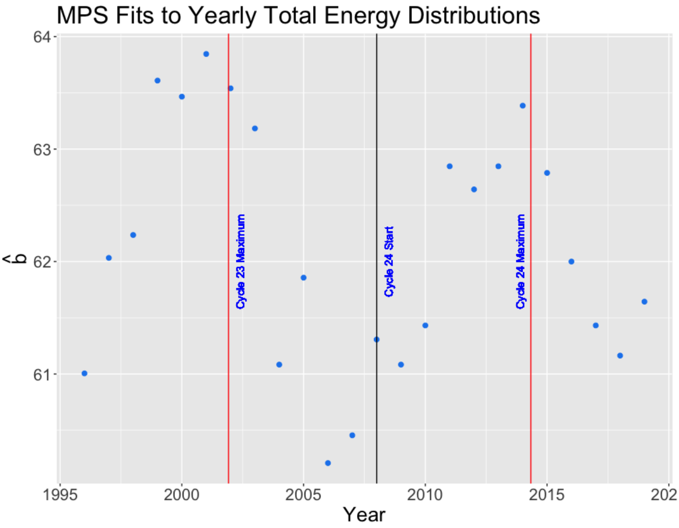
MPS Fit to Aggregate Total Energy



MPS α Fits to Yearly Total Energy



MPS Left Bound Fits to Yearly Total Energy



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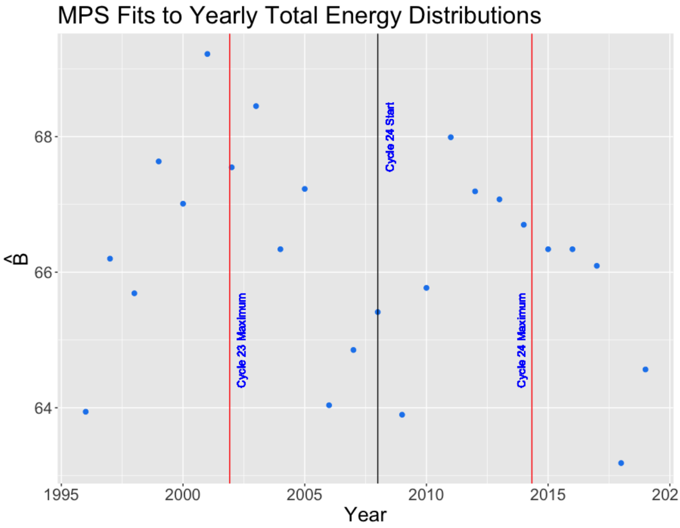
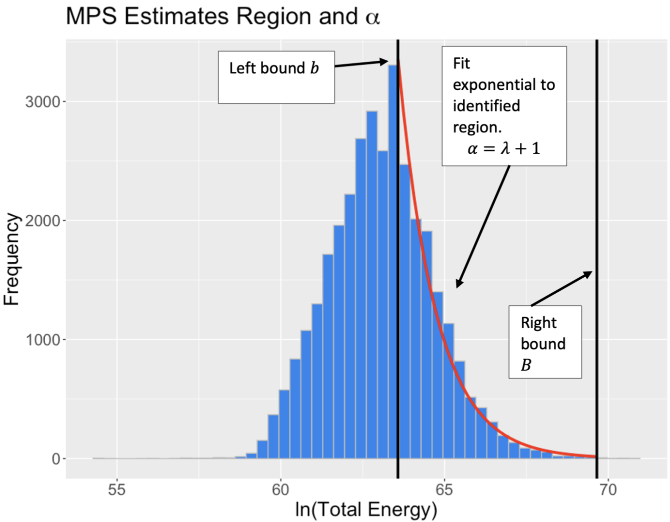


MPS Fit Method



Semi-Parametric Multinomial Model