Functional summary statistics for Poisson processes on convex shapes in three dimensions

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My Motivation

- Pseudomonas aeruginosa is a virulent and opportunistic bacteria
- A major cause of hospital borne infections and a serious danger to patients undergoing immunosuppressive therapy
- P. aeruginosa virulence is, in part, attributed to its type 6 secretion system (T6SS)
- The T6SS originates on the cell membrane of P. aeruginosa and we want to understand its spatial distribution with respect to its shape.
- Is the point pattern completely spatially random (CSR) or is there preferential placement where the T6SS builds and activates?

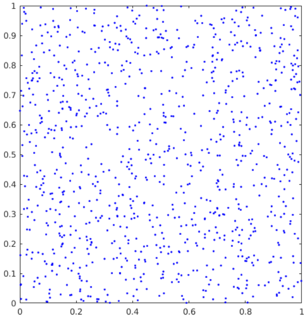
Spatial point pattern analysis in $\mathbb{R}^2$

- A spatial point process is the stochastic mechanism that gives rise to a point pattern
- Intensity measure and function of a process,

$$\mu(B) \equiv \mathbb{E}[N(B)] = \int_B \rho(\mathbf{x}) d\mathbf{x}.$$

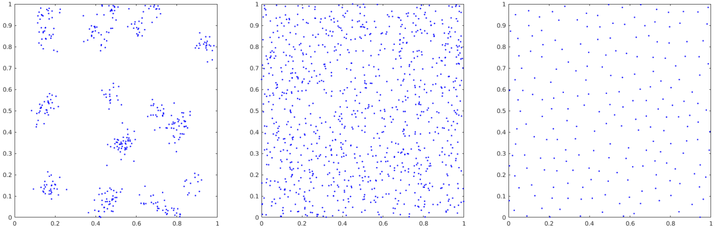
- A process is homogeneous if ρ is constant.
- A process is stationary if it is distributionally invariant to translations.

Complete spatial randomness on $\mathbb{R}^2$



- Poisson process with constant intensity function, ρ.
- Poisson number of points uniformly and independently distributed.

Typical point patterns



(a) Cluster (b) Complete Spatial Randomness (c) Regular

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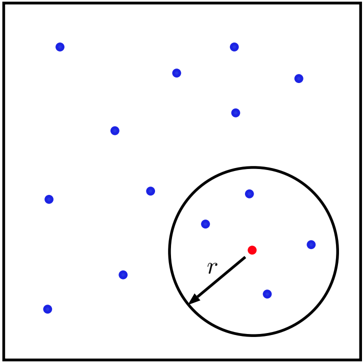
Summary statistics for planar and spatial data

- Determining whether a point pattern exhibits CSR is normally the first step.
- This is commonly achieved using functional summary statistics, such as nearest neighbour function.
- Based on these deviations we can suggest whether a pattern is more regular or clustered.

Ripley's K-Function



- A popular functional summary statistic is Ripley's K-function,

$$K(r) = \frac{1}{\rho}\mathbb{E}[N_{\mathbf{0}}(B(\mathbf{0}, r))],$$

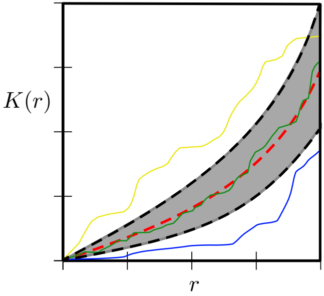
- For a homogeneous Poisson process $K(r) = \pi r^2$.

- Estimator is,

$$\hat{K}(r) = \frac{\text{Area}^2(B)}{N(N-1)} \sum_{\mathbf{x},\mathbf{y} \in X \cap B}^{\neq} \mathbb{1}[d(\mathbf{x}, \mathbf{y}) \leq r].$$

Interpretation of Ripley's K-function

- CSR: $\hat{K}(r) \approx \pi r^2$
- Cluster: $\hat{K}(r) > \pi r^2$
- Regular: $\hat{K}(r) < \pi r^2$



Inhomogeneous K-Function

- Ripley's K-function can be extended to a class of inhomogeneous processes [1].
- For this class of inhomogeneous processes the K-function is,

$$K_{\text{inhom}}(r) = \mathbb{E} \sum_{\mathbf{x} \in X^{!}_{\mathbf{y}}} \frac{\mathbb{1}[\mathbf{x} \in B(\mathbf{y}, r)]}{\rho(\mathbf{x})},$$

- Estimators take the form,

$$\hat{K}_{\text{inhom}}(r) = \frac{1}{\text{Area}(B)} \sum_{\mathbf{x},\mathbf{y} \in X \cap B}^{\neq} \frac{\mathbb{1}[\mathbf{x} \in B(\mathbf{y}, r)]}{\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y})},$$

Statement of Problem

Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$*, with* $N = |X|$*, be a spatial point pattern on a convex space in* $\mathbb{R}^3$*,* $\mathbb{D}$*. Then we wish to determine if,*

$$H_0 : X \text{ is CSR on } \mathbb{D} \qquad H_1 : X \text{ is not CSR on } \mathbb{D}$$

Point processes on the unit Sphere

- The metric on the unit sphere is the great circle length,

$$d(\mathbf{x}, \mathbf{y}) = \cos^{-1}(\mathbf{x} \cdot \mathbf{y}).$$

- A point process is said to be isotropic on the sphere if it's distribution is invariant under rotations [2].
- Poisson processes with intensity function, $\rho : \mathbb{S}^2 \mapsto \mathbb{R}$, on a sphere is defined
  1. Poisson$(\mu(\mathbb{S}^2))$ number of points
  2. Given the number of points, these points are independently distributed with density proportional to $\rho(\mathbf{x})$ on $\mathbb{S}^2$.

K-Function on sphere for Poisson processes

- K-function for Poisson process is,

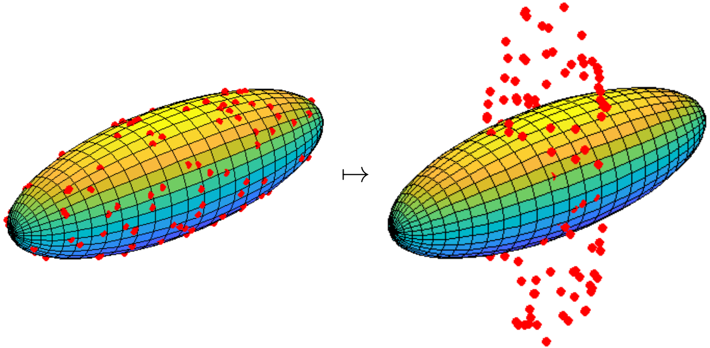
$$K(r) = K_{\text{inhom}}(r) = 2\pi(1 - \cos r).$$

- Non-parametric estimator of $K_{\text{inhom}}(r)$ is,

$$\hat{K}_{\text{inhom}}(r) = \frac{1}{4\pi} \sum_{\mathbf{x},\mathbf{y} \in X}^{\neq} \frac{\mathbb{1}[d(\mathbf{x},\mathbf{y}) \leq r]}{\rho(\mathbf{x})\rho(\mathbf{y})}, \tag{1}$$

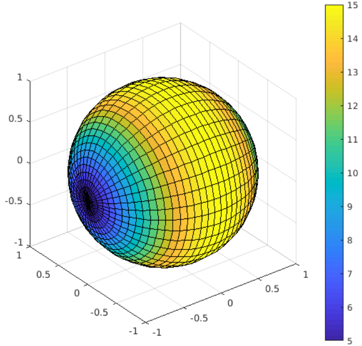
Main challenge for point pattern analysis on convex shapes

- K-functions rely on definitions of stationarity/isotropy.
- For a general convex shape these definitions do not extend.



Mapping homogeneous Poisson processes from an ellipsoid to the unit sphere

- Mapping Theorem [3]: A Poisson process is invariant under transformations between metric spaces.
- Use $(x, y, z) \mapsto (x/a, y/b, z/c)$
- The intensity function becomes



$$\rho^*(\mathbf{x}) = \rho ab \left[1 - \left(1 - \frac{c^2}{a^2}\right)\mathbf{x}_1^2 - \left(1 - \frac{c^2}{b^2}\right)\mathbf{x}_2^2\right]^{\frac{1}{2}}.$$

Constructing K-functions for Poisson processes on ellipsoids

- We can construct $\hat{K}_{\text{inhom}}(r)$ as,

$$\hat{K}_{\text{inhom}}(r) = \frac{1}{4\pi(\rho ab)^2} \sum_{\mathbf{x}\in X} \sum_{\mathbf{y}\in X\setminus\{x\}} \frac{\mathbb{1}[d(\mathbf{x},\mathbf{y})]}{\tilde{\rho}(\mathbf{x})\tilde{\rho}(\mathbf{y})},$$

where $\tilde{\rho}(\mathbf{x}) = \left[1 - \left(1 - \frac{c^2}{a^2}\right)\mathbf{x}_1^2 - \left(1 - \frac{c^2}{b^2}\right)\mathbf{x}_2^2\right]^{\frac{1}{2}}$.

- Use the following unbiased estimator for ρ^2,

$$\hat{\rho}^2 = \frac{N(N-1)}{(\text{Area of ellipsoid})^2}$$

Test statistic for complete spatial randomness

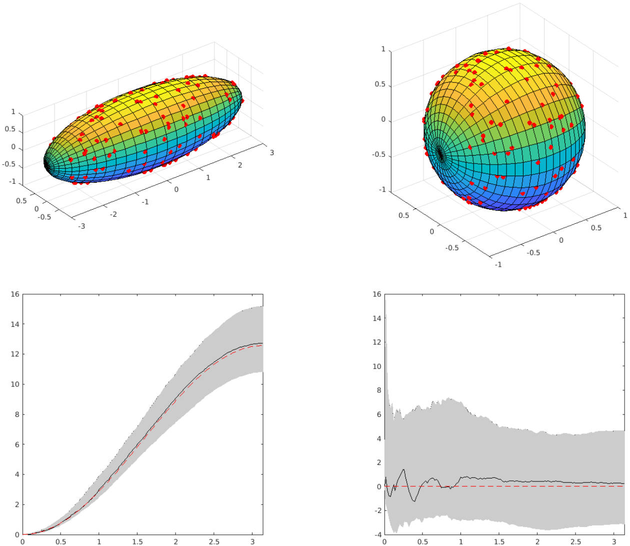
- In $\mathbb{R}^2$, the L-function is used to determine CSR,

$$L(r) = \left(\frac{K(r)}{\pi}\right)^{\frac{1}{2}} = r,$$

  as it is variance stabilised.
- We derive $\mathrm{Var}(\hat{K}_{\mathrm{inhom}}(r))$ and standardise $\hat{K}_{\mathrm{inhom}}(r)$ for each r to stabilise variance [4].
- We suggest using the following test statistic,

$$T = \sup_{r \in [0,\pi]} \left| \frac{\hat{K}_{\mathrm{inhom}}(r) - 2\pi(1 - \cos r)}{\sqrt{\mathrm{Var}(\hat{K}_{\mathrm{inhom}}(r))}} \right|$$

Example: Homogeneous Poisson process

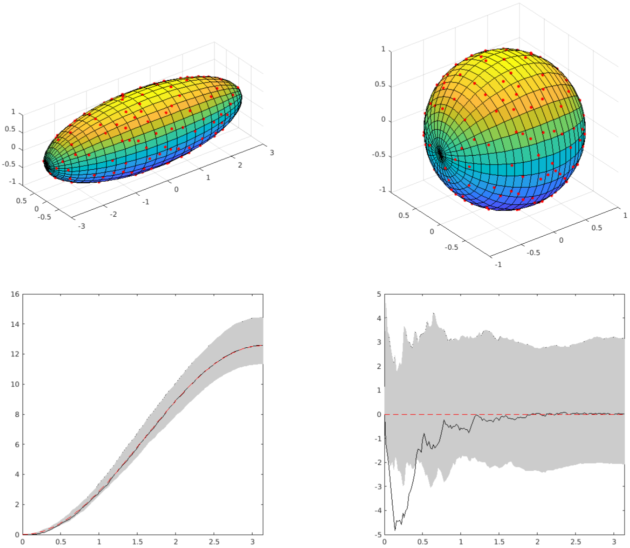


Example: Regular process

We simulate the Matèrn 2 process on ellipsoids.

1. Simulate a Poisson process with constant intensity function ρ.
2. For each point, simulate a *mark*, from a mark distribution independently of the locations of all the points and each other
3. For any two points within a distance R of each other remove the one with the smaller mark.

Example: Regular process

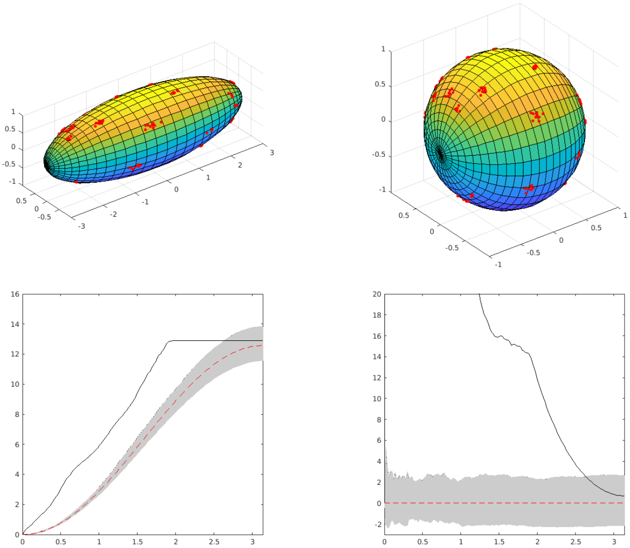


Example: Cluster process

We simulate a *Thomas* type ellipsoid process.

1. Simulate a parent Poisson process with constant intensity ρ.
2. For each parent draw N Poisson random variable offspring.
3. Independently distribute each offspring with a von Mises Fisher distribution centred at the parent point.

Example: Cluster process



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Interpretation of K-function ECF

- From the previous plots we see that,
  - Regular processes: $K(r)$ is below the lower simulation envelope for small values of r
  - Cluster processes: $K(r)$ is larger than the upper simulation envelope for many values of r
- This coincides with the interpretation of $K(r)$ for planar and spatial data.

H-, F-, and J-function

- The K-function only captures part of the information within an observed spatial point pattern.
- Other functional summary statistics include,

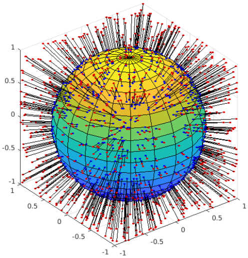
$$F(r) = 1 - P_{X \cap B(\mathbf{0},r)}(\emptyset),$$
$$H(r) = 1 - P_{X_{\mathbf{x}}^{!} \cap B(\mathbf{0},r)}(\emptyset),$$
$$J(r) = \frac{1 - H(r)}{1 - F(r)},$$

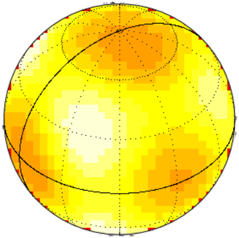
General convex shapes

- We can use,

$$f(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2}}, \qquad (2)$$

which maps each point of the space uniquely to the sphere.





Potential astrophysics applications

- This methodology is not restricted to convex shapes within $\mathbb{R}^3$ but can be applied to convex shapes in any dimension.
- In particular it could be extended for point patterns on ellipses.
- Consider an event on a planet. Can we determine if these events occur more frequently when closer to one of the foci of it's orbit?
- For example do natural disasters occur more when the Earth is closer to the Sun?
- Are the other potential applications?

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