Luminosity Functions

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Introduction: What is a Luminosity Function?

Figure: A galaxy cluster.
Introduction: Project Goal

- The Luminosity Function specifies the relative number of sources at each luminosity.
Introduction: Project Goal

- The Luminosity Function specifies the relative number of sources at each luminosity.
- Goal of the Project: To develop a fully Bayesian model to infer the distribution of the luminosities of all the sources in a population.
Introduction: Data

- $Y_i$, observed photon counts, contaminated with background in a source exposure.
- $X$, observed photon counts in the exposure of pure background.
Bayesian Model

- **Level I model:**
  \[ X | \xi \sim \text{Pois}(\xi), \]
  \[ Y_i = Y_{iB} + Y_{iS}, \text{ where } Y_{iB} | \xi \sim \text{Pois}(a_i \xi), \]
  \[ Y_{iS} | \lambda_i \sim \begin{cases} \delta_0, & \text{if } \lambda_i = 0; \\ \text{Pois}(b_i \lambda_i), & \text{if } \lambda_i \neq 0. \end{cases} \]

- $\xi$ is the background intensity,
- $\lambda_i$ is the intensity of source $i$,
- $a_i$ is ratio of source area to background area (known constant),
- $b_i$ is the telescope effective area (known constant).
Bayesian Model

- Level II model:

\[ \xi \sim \text{Gamma}(\alpha_0, \beta_0), \]
\[ \lambda_i \mid \alpha, \beta, \pi \begin{cases} = 0, & \text{with probability } 1 - \pi; \\ \sim \text{Gamma}(\alpha, \beta), & \text{with probability } \pi. \end{cases} \]

- Level III model:

\[ P(\alpha, \beta, \pi) \propto \frac{1}{\beta^3} \pi^{c_1 - 1} (1 - \pi)^{c_2 - 1}. \]
Bayesian Model: Summary

- **Model**

  \[ X|\xi \sim \text{Pois}(\xi), \ Y_i = Y_{iB} + Y_{iS}, \]
  \[ Y_{iS}|\xi \sim \text{Pois}(a_i\xi), \ Y_{iB}|\lambda_i \sim \text{Pois}(b_i\lambda_i), \]
  \[ \xi \sim \text{Gamma}(\alpha_0, \beta_0), \]
  \[ \lambda_i|\alpha, \beta, \pi \begin{cases} = 0, & \text{with probability } 1 - \pi, \\ \sim \text{Gamma}(\alpha, \beta), & \text{with probability } \pi, \end{cases} \]
  \[ P(\alpha, \beta, \pi) \propto \frac{1}{\beta^3}. \]

- **Research interest:**
  - The posterior distribution of intensities \( \lambda \),
  - The posterior distribution of \( 1 - \pi \), the proportion of dark sources.
Simulation Study 1: Setup

- Number of sources, $N=500$.
- Distribution of $\lambda$’s:
  
  \[
  \lambda_i \overset{iid}{\sim} \begin{cases} 
  \delta_0, \text{ with prob } 1 - \pi = 0.15, \\
  \text{Gamma}[10, 35] \text{ with prob } 0.85.
  \end{cases}
  \]
- Distribution of $Y_{iS}$:
  \[
  Y_{iS} | \lambda_i \sim \text{Pois}(b_i \lambda_i).
  \]
- Distribution of background noise $Y_{iB}$:
  \[
  Y_{iB} | \xi \sim \text{Pois}(4), \text{ approximately}.
  \]

\[
Y_i = Y_{iS} + Y_{iB}.
\]
Posterior Distributions of the Hyper-parameters

- Posterior draws of alpha/beta
- Posterior draws of alpha/beta^2
- Posterior draws of 1-pi
Posterior Distributions of the Hyper-parameters

Scatter Plots of Posterior Draws

Joint Posterior Density of $\pi$ and $\alpha/\beta$
Distribution of source intensities

\[ \lambda_i \overset{iid}{\sim} \begin{cases} \delta_0, & \text{with prob } 1 - \pi, \\ \text{Gamma}(\alpha, \beta), & \text{with prob } \pi. \end{cases} \]
Simulation Study 2: Setup

- Number of sources, $N=500$.
- Distribution of $\lambda$’s:
  \[ \lambda_i \overset{iid}{\sim} \begin{cases} \delta_0, & \text{with prob } 1 - \pi = 0.15, \\ \text{Gamma}[2, 60] & \text{with prob } 0.85. \end{cases} \]
- Distribution of $Y_i S$:
  \[ Y_i S | \lambda_i \sim \text{Pois}(b_i \lambda_i). \]
- $Y_{iB} | \xi \sim \text{Pois}(2)$, approximately.

![Observed Photon Counts](image-url)
Simulation Study 2

- Posterior draws of alpha/beta
- Posterior draws of alpha/beta^2
- Posterior draws of 1-pi

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Simulation Study 2

Scatter Plots of Posterior Draws

- The True Value
- Posterior Draws

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Simulation Study 2

Distribution of source intensities
Choices of Priors for the Hyper-parameters

- Recall

\[ \lambda_i | \alpha, \beta, \pi \overset{iid}{\sim} \begin{cases} \delta_0, \\ \text{Gamma} \left[ \frac{\alpha}{\beta}, \frac{\alpha}{\beta^2} \right] = \text{Gamma} \left[ \mu, \frac{\mu}{\beta} \right] \end{cases} \]

with prob $1 - \pi$, with prob $\pi$. 

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Choices of Priors for the Hyper-parameters

- Recall

\[ \lambda_i | \alpha, \beta, \pi \sim \begin{cases} 
\delta_0, & \text{with prob } 1 - \pi, \\
\text{Gamma}[\frac{\alpha}{\beta}, \frac{\alpha}{\beta^2}] = \text{Gamma}[\mu, \frac{\mu}{\beta}], & \text{with prob } \pi.
\end{cases} \]

- 

\[ P(\mu, \beta, \pi)d\mu d\beta d\pi \propto P(\beta)P(\pi)d\mu d\beta d\pi \propto \frac{1}{\beta} P(\beta)P(\pi)d\alpha d\beta d\pi, \]

\[ P(\alpha, \beta, \pi)d\alpha d\beta d\pi \propto \frac{1}{\beta^{c_3+1}} \pi^{c_1-1} (1 - \pi)^{c_2-1} d\alpha d\beta d\pi \]
Choices of Priors: Simulation Study

- \( \pi \sim \text{Beta}(c_1, c_2) \), we can use informative prior if we have some prior information about the distribution of \( \pi \). Otherwise, we can let \( c_1 = c_2 = 1 \), so \( \pi \sim \text{Unif}(0, 1) \).

- Priors for \((\alpha, \beta)\):
  
  Prior 0: \( P(\alpha, \beta) \propto 1 \),
  
  Prior 1: \( P(\alpha, \beta) \propto \frac{1}{\beta} \),
  
  Prior 2: \( P(\alpha, \beta) \propto \frac{1}{\beta^2} \),
  
  Prior 3: \( P(\alpha, \beta) \propto \frac{1}{\beta^3} \).
Choices of Priors: Coverage for $\pi$

Graphs showing the coverage for different prior distributions and gamma distributions.

- Prior 0
- Prior 1
- Prior 2
- Prior 3

Distributions:
- Gamma[2, 60]
- Gamma[4, 100]
- Gamma[8, 200]
- Gamma[2, 20]
- Gamma[4, 30]
- Gamma[8, 50]
- Gamma[2, 2]
- Gamma[4, 5]
- Gamma[8, 8]
Choices of Priors: Coverage for $\frac{\alpha}{\beta}$

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Choices of Priors

- Conclusion: the prior

\[ P(\alpha, \beta) \propto \frac{1}{\beta^3} \]

gives the highest frequency coverage in most simulation studies.

- This prior is called Stein’s Harmonic Prior. The SHP prior is shown to provide estimators that have adequate frequency coverage.
It takes about 3 hours to get 100,000 draws.

Metropolis-Hastings algorithm within Gibbs Sampler:

\[ P(\alpha|\beta, \pi, \xi, \lambda, Y_B, X, Y) \propto \left( \frac{\beta \lambda^*}{\Gamma(\alpha)} \right)^K, \]

where \( K = \sum 1_{\lambda_i \neq 0} \), and \( \lambda^* = (\prod_{i, \lambda_i \neq 0} \lambda_i)^{1/K} \).

Is there a good way to sample from the conditional posterior distribution?
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