Statistical Anaysis of Fe Lines

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Statistical Model

- Suppose we observe the intensity of lines of wavelength $\Lambda = \{\lambda_1, \ldots, \lambda_J\}$.
- Let $\hat{I}_j$ be the observed intensity in λ_j, and $\hat{\sigma}_j$ its error.
- We then model the data as

$$\hat{I}_j \sim \text{Normal}\left(A_j(n_e, T_e)n_e^2 \mathrm{d}s, \hat{\sigma}_j\right),$$

  where A_j is the atomic data curve for line λ_j, n_e is the electron density, T_e is the electron temperature, and ds is the path length through the solar atmosphere.
- We assume that all of the Fe XIII emission is formed at the same temperature and use a fixed T_e.

Model Fitting 1: Pragmatic Bayes

- Let $Y = \left\{\hat{I}_1, \ldots, \hat{I}_J\right\}$ and $\Theta = \{n_e, ds\}$.
- Denote the prior distribution on the atomic data curve by $P(A)$, the joint prior distribution on n_e and ds by $P(\Theta)$, and the likelihood function by $P(Y \mid \theta, A)$.
  - We use flat prior distributions on $\log_{10}(n_e)$ and $\log_{10}(ds)$.
- **Pragmatic Bayes:** Assume that $P(A \mid Y) = P(A)$ such that

$$P(\Theta, A \mid Y) = P(\Theta \mid A, Y) P(A)$$

  - For $l = 1, \ldots, L$:

    Step 1: Sample A_l from its prior distribution: $A_l \sim P(A)$.

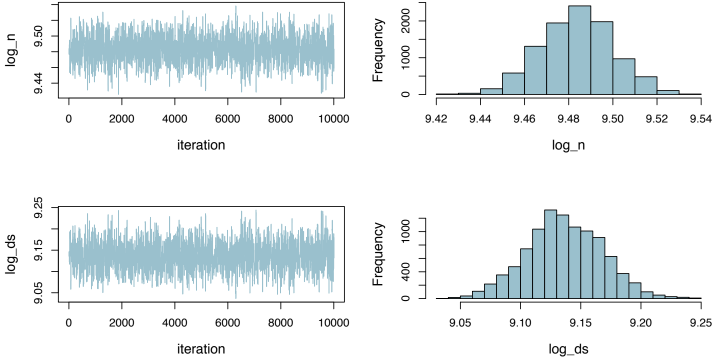
    Step 2: For $k = 1, \ldots, K$: sample $\Theta^{(k)}$ from its joint posterior distribution given A_l :

$$\Theta^{(k)} \sim P(\Theta \mid Y, A_l).$$

MCMC Methodology

- Use a Metropolis algorithm to explore $P(\Theta \mid Y, A_l)$.
- At iteration $l+1$, draw $\Theta^{(*)} \sim \text{Normal}\left(\Theta^{(l)}, \left[(2.38)^2/2\right] \Psi\right)$.[1]
  - Ψ is an estimate of the posterior variance-covariance matrix computed via the Laplace approximation.

MCMC for exploring P(ΘlY, A$^{(1)}$)



[1] See Gelman et al. (1996) for details on choice of proposal dist'n.

Model Fitting 2: Full Bayes

- **Full Bayes:** $P(\Theta, A \mid Y) = P(\Theta \mid A, Y)\, P(A \mid Y)$.

- The fully Bayesian posterior distribution on Θ can be thought of as a weighted version of that under Pragmatic Bayes:

$$P(\Theta \mid Y) = \int P(\Theta \mid A, Y)\, \frac{P(A \mid Y)}{P(A)} P(A)\, \mathrm{d}A.$$

- Alternative computational techniques are used to obtain samples from $P(\Theta \mid Y)$.

Full Bayes: Mathematical Details

- Let $\Phi(\Theta)$ be an approximation to $P(\Theta \mid A, Y)$
- Notice $\frac{P(A \mid Y)}{P(A)} P(Y) = \int \frac{P(Y \mid \Theta, A) P(\Theta)}{\Phi(\Theta)} \Phi(\Theta) d\Theta$
- With $w(A) = \frac{P(A \mid Y)}{P(A)}$,

$$w(A) P(Y) \approx \frac{1}{L} \sum_{l=1}^{L} \frac{P\left(Y|\Theta^{(k)},A\right)P\left(\Theta^{(k)}\right)}{\Phi\left(\Theta^{(k)}\right)},$$

where $\Theta^{(k)} \sim \Phi(\Theta)$.Now,

$$P(\Theta \mid Y) = \int P(\Theta \mid A, Y) \frac{P(A \mid Y)}{P(A)} P(A) dA.$$

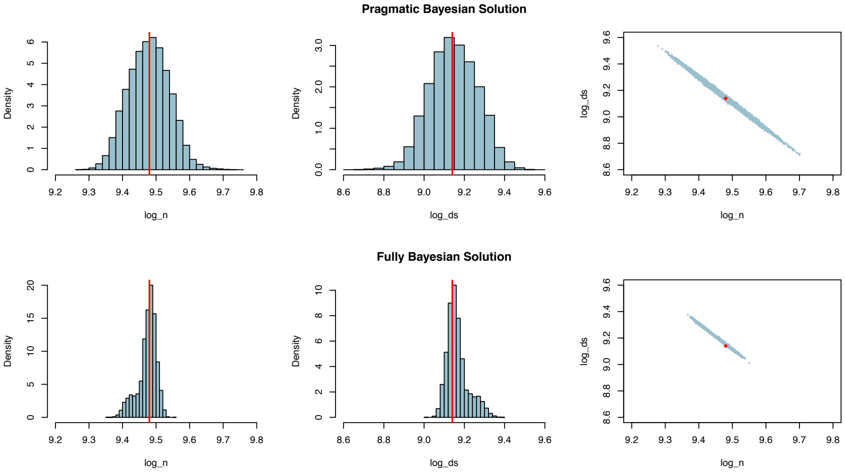
Full Bayes: Mathematical Details

- Let $\tilde{w}(A) = \dfrac{w(A)}{\max w(A)}$, so max $\tilde{w}(A) = 1$.
- For each A_l we have a sample from $P(\Theta \mid A_l, Y)$, call them $\left\{\Theta_l^{(1)}, \ldots, \Theta_l^{(K)}\right\}$.
- For each of these samples, let $z_l^{(k)} \sim \text{Bernoulli}(\tilde{w}(A_l))$.
- Then, keep all the $\Theta_l^{(k)}$ such that $z_l^{(k)} = 1$. This new sample is an approximate sample from $P(\Theta \mid Y)$.
- Note that the joint dist'n of A, Θ, Z is

$$P(A)\,P(\Theta \mid Y, A)\left[h\frac{P(A \mid Y)}{P(A)}\right]^{z}\left[1 - h\frac{P(A \mid Y)}{P(A)}\right]^{1-z}$$

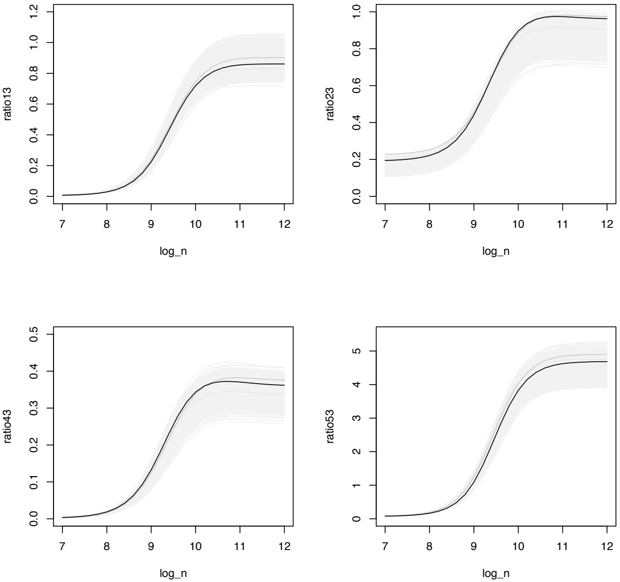
where h is a constant up to Y.

Pragmatic vs. Full Bayes

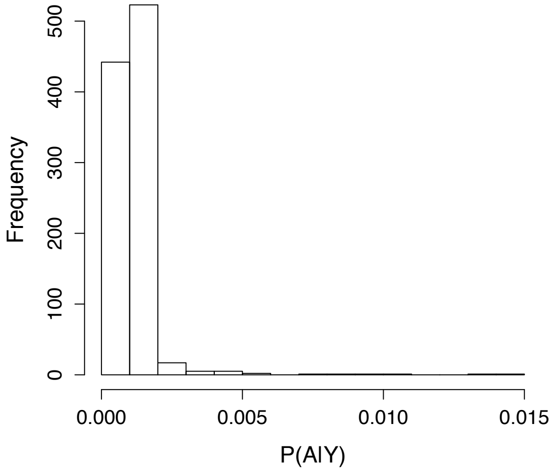


Pragmatic vs. Full Bayes Solutions. Red lines and diamonds are represent the chi-squared fit.

Weighted Atomic Data Curves



$P(A \mid Y)$



Thanks!