Improving the Analysis of Solar and Stellar Observations

Applications of Bayesian Analysis to Coronal Seismology

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Abstract: The objective of this document and the accompanying slides is to show examples of the application of Bayesian analysis tools to the inversion of physical parameters in coronal seismology. Coronal seismology refers to the methods developed to probe solar plasmas by comparison between theoretically predicted magnetic wave properties with those observed. In the last years, we have been developing several techniques to perform parameter inference, model comparison, and model averaging using waves and oscillations in magnetic and plasma structures of the solar corona. The idea is to show some of these results to explain some Bayesian concepts that might be applicable to our problems and see if they can help.

1. The title of my talk is “Applications of Bayesian analysis to coronal seismology”. The idea is to show examples of the application of Bayesian analysis to the inversion of physical parameters in coronal seismology, to see if this can be of help to our problems.

2. There are some things that I know to be true, e.g., “my name is Iñigo Arregui” and others that I know to be false, “I am not Harry Warren”. There remain many things whose truth of falsity are unknown to me. I could say that I am uncertain about them. Many of those are related to the Sun and, in particular, to the solar atmosphere.

3. Truth and falsity are the subjects of logic, which has a long history going back to Aristotle in the Western world. The study of uncertainty is much more recent. We understand probability as extended logic, a tool to quantify uncertainty in terms of degree of belief.

4. My research area is called seismology of the solar atmosphere. The aim is the determination of difficult to measure physical parameters in the solar atmosphere, by a combination of observed and theoretical properties of magnetic waves that are present in structures such as coronal loops or prominences.

5. The existence of waves and oscillations in magnetic and plasma structures of the solar atmosphere is now beyond question. Early observations, already in the 70s, pointed to the existence of quasi-periodic perturbations in solar coronal structures. The detection of these oscillations was mainly based on the measurement of the temporal and spatial variation of spectroscopic properties (such as intensity, line width, and Doppler velocity) of coronal emission lines. The recent high resolution imaging and spectroscopic observations have enabled us to measure these waves with increasing detail. Wave-like dynamics is now found in different regions of the solar atmosphere in structures with different physical properties.

6. The method of MHD seismology works as follows. We have observation of solar atmospheric magnetic structures from which a number of physical parameters, such as the temperature, density, or magnetic field strength, are unknown. We can use those observations to propose theoretical models. Very often, observations also show the existence of waves and oscillations in the observed structures. We can then study the theoretical properties of MHD waves in the modelled structures and compare them to the observed wave properties. By gradually improving theory by learning from observations one expects to improve models and at the same time determine the unknown physical parameters.

7. The method is akin to those developed to study e.g., the interiors of the Earth or the Sun using waves; the physical conditions of magnetospheric plasmas, accretion disks around compact objects; or fusion plasmas in tokamak experiments.

8. This is a typical problem in which probabilistic inference has to be applied. The reason is that we need to solve an inverse problem. In the forward problem, we prescribe theoretical models and parameters (the causes) and analyse the theoretical wave properties (the consequences). In the inverse problem, we try to infer the causes (the unknown physical parameters/models) from the consequences (the observed wave properties). This problem has to be solved under conditions in which information is incomplete and uncertain. For these reasons, we use the rules of probability to make scientific inference and quantify uncertainty.

9. We are going to make statements in terms of probability and this is a slide to clarify what we mean by this word. Probability is a tool to quantify randomness and (as is our case) uncertainty. Statistics uses this tool to make scientific inference. There are two main schools/lines of thought/religions in the use of probability. They are both correct and useful, but they calculate probabilities of different things. Frequentists measure occurrence rates or frequencies. Probability for them is the long-run relative frequency in the limit of infinite repetitions of e.g., the same experiment. They focus on alternative data and compare the occurrence rate of different data realisations. This is useful for counting or characterising data. Bayesians measure informed belief. Probability for them is a measure of the degree to which a given proposition is supported by data. They focus on alternative hypotheses and compare probabilities of different hypotheses in view of data. This is useful and necessary for inference and model comparison. Astrophysics (solar physics) is an observational science. Data are fixed. Hence the Bayesian framework is the only way we have to perform parameter inference and model comparison, to see how data constrain parameters and models, and to propagate uncertainty from data to inferred parameters. The framework defines rigorous tools to do all this.

10. In my view, we cannot state that something is true or false in the solar atmosphere. We just try to quantify what to believe concerning physical parameters and models. And accept that as the best we can do. I suspect this is applicable to most of the astrophysical research.

11. Bayesian analysis considers any inversion problem, in terms of probabilistic inference, as the task of estimating the degree of belief on statements about parameter values or model evidence, conditional on observed data. It uses Bayes' rule, which says that the state of knowledge is a combination of what we know independently of the data (the prior) and the likelihood of obtaining a given data realisation as a function of the parameter vector (the likelihood function). This gives the posterior distribution that accounts for what can be said about a parameter or model, conditional on data. Bayes' rule can be applied to the problems of parameter inference, model comparison, and model averaging. In parameter inference the posterior is computed for different combinations of parameters and then one marginalises to obtain information about the one in which we are interested. In model comparison, the ratio of posteriors for alternative models is computed to assess which one better explains observed data. Finally, model averaging enables us to combine the alternative posteriors by weighting them with the evidence for each model.

12. The following is a list of methodologies and applications we have developed in the last years for the application of Bayesian analysis to coronal seismology. Applications include the inference of physical parameters in oscillating magnetic structures in coronal and prominence plasmas using observed properties of damped transverse oscillations; the inference and model comparison of the coronal density scale height and magnetic field expansion, using multiple mode period oscillations; or the model comparison for the density structure along and across coronal waveguides, using the properties of transverse kink waves. The employed methodologies include: the MCMC sampling of posterior distributions; the computation of marginal posteriors from the integrals of the full posterior; the computation of the marginal

likelihood to assess model plausibility; the computation of Bayes factors to assess relative model plausibility; or the calculation of weighted posteriors to perform model averaging.

13. Our first example deals with the determination of physical parameters in oscillating coronal loops.

14. Transverse oscillations of coronal loops are well-known since they were discovered by TRACE. After a large scale, impulsive, perturbation in the corona, such as a flare or filament eruption, individual or groups of coronal loops are seen to oscillate in their transverse direction. These oscillations have periods of a few minutes. An important property is that oscillations are quickly damped in time, with damping time scales of a few oscillatory periods only. Transverse loop oscillations were interpreted as the fundamental standing fast MHD kink mode of a magnetic flux tube, the only oscillatory mode that produces the lateral displacement of an axisymmetric flux tube. Regarding the damping, several mechanisms have been proposed. The one that seems to better explain the observations is resonant damping. Because of the inhomogeneity of the medium in the transverse direction, the global kink mode is coupled to local Alfvén waves and energy is transferred to motions at the boundary of the tube.

15. In coronal loops, the forward problem is reduced to the solution of two algebraic equations for the period and damping ratio of resonantly damped kink oscillations in 1D magnetic flux tubes under the thin tube and thin boundary approximations. When no further assumptions are made, these wave properties are functions of three unknown parameters: density contrast, transverse inhomogeneity length-scale, and internal Alfvén travel time. The classic inversion technique simply consists of imposing these two functions to be equal to the observed periods and damping times. As we have two observables and three unknowns, there is an infinite number of equally valid equilibrium models that explain observations. However, these solutions must follow a particular 1D solution space in the 3D parameter space.

16. This solution curve was first obtained by Arregui et al. (2007), numerically, and then by Goossens et al. (2008), analytically. The figure shows the valid equilibrium models that reproduce observed period and damping rates in the three-dimensional parameter space of density contrast, transverse inhomogeneity, and Alfvén travel time. Although there are in principle infinite possibilities, the Alfvén travel time is found to be constrained to a rather narrow range. We see an excellent agreement between the analytic inversion (solid lines) and the numerical inversion (dots) outside the thin tube and thin boundary approximations. This is a general solution, but has two main limitations. First, the fact that there is an infinite number of equally valid solutions. Also, it is not clear how to propagate errors from observations to inferred parameters.

17. Four years ago, we followed a completely different approach using the Bayesian framework. We computed the posteriors using different prior information for the three unknowns and a Gaussian likelihood for data.

18. We found that the optimal result is obtained when some additional information on density contrast is introduced. In that case, data is able to fully constrain the three unknowns.

19. Marginal posteriors for the three parameters are obtained from which estimates can be obtained with correct propagation of uncertainties from data to inferred parameters. The figures show the marginal posteriors for the three unknowns and the joint probability distribution for l/R and Alfvén travel time with 68 and 95 % confidence intervals.

20. We applied the technique to 11 events of loop oscillations and compared the inversion results to previous analytical constraints.

21. Our second example deals with the use of observed spatial damping of coronal waves for the determination of the cross-field density structure of magnetic waveguides.

22. Propagating transverse waves in the corona also show spatial damping. This has been found by Tomczyk et al. (2007) and Tomczyk & McIntosh (2009). These movie shows Doppler velocities measured with the CoMP instrument. The motions show almost zero compressibility, no intensity oscillations, hence they were interpreted as Alfvén waves. There is a discrepancy between inward/outward power in these waves that points to the possibility of in situ damping. Indeed, resonant damping predicts a frequency dependent attenuation of wave motions and observational/theoretical results seem to point to this mechanism being responsible for the observed damping.

23. The spatial damping of propagating kink waves has been extensively studied both analytically and numerically. As can be clearly seen in the simulations by Pascoe et al. for propagating kink waves resonant absorption produces the attenuation of wave amplitude in space.

24. The decay was thought to have an exponential profile along the waveguide until, trying to fit the amplitude computed in numerical simulations, Pascoe et al. found that a Gaussian profile better fits the initial damping stage. Analytical studies for the spatial damping by Hood et al. and for time damping by Ruderman & Terradas have recently shown that the existence of two different damping regimes (Gaussian + exponential) is an inherent feature of resonant absorption. The exponential profile only accounts for the asymptotic behavior in space/time. Analytical expressions for the Gaussian, exponential damping length scales and the position at which the damping regime changes have been obtained as a function of the two parameters that define the cross-field density structuring of the waveguide. This means we have additional information without the need to include new parameters.

25. In our analysis we have considered the inference of the cross-field density structuring from measurements of the Gaussian damping length and the height of change of damping regime. The forward model is analytical and given by these expressions for Lg and h. We have generated synthetic data for parameter values in the expected and reasonable parameter space. Then, a Gaussian likelihood and uniform prior distributions for the contrast and transverse inhomogeneity length scale are considered. The inference is performed by using Bayes' rule and marginalizing.

26. The top two figures show the marginal posteriors for contrast and inhomogeneity length scale. They show well peaked probability distributions. The bottom figure shows the joint posterior with 68 and 95 % credible intervals. The existence of two damping regimes enables us to fully constrain the cross-field density structuring.

27. We have repeated the inversions for different values of parameters using the analytical forward model. Overall we recover the input parameters correctly. We have also performed numerical simulations of the propagation and damping processes. After fitting the signal along the waveguide, numerical values for the damping length scales are obtained and the inversion is repeated. We see that the analytical forward model is an accurate representation of the full numerical solutions. Also, large density contrast produce the largest errors, since in those cases the Gaussian stage of the damping is shorter, sometimes comparable to the wavelength and the fitting is problematic. These cases represent a challenge from the observational point of view.

28. There is an easy alternative way of obtaining information about the cross-field density structure in coronal waveguides from the damping of transverse wave by making use of the definition of joint probability and marginal posteriors

29. Imagine your forward problem is simply that the number c is the product of a and b. We can construct the joint probability of a and b, given c, which is shown in this surface plot. Then, for a fixed value of a, p(b|a,c) is this dashed line here which gives us the probability of b, given a and c. If we wish the probability of b, given c, all we have to do is to integrate for all values of a. For a fixed value of b, p(a|b,c) is this solid line here which gives us the probability of a, given b and

c. If we wish the probability of b, given c, all we have to do is to integrate for all values of a. Using this simple method, we can solve our problem.

30. The damping ratio of transverse oscillations is a function of two parameters: the density contrast and the transverse inhomogeneity length-scale. We can infer both of them using Bayes theorem assuming uniform priors in given ranges and a Gaussian likelihood function. Then, one constructs a joint probability distribution as the one shown before for the product of two numbers and marginalises.

31. The results show well-constrained posteriors for both unknowns, although a long tail is present for the density contrast which implies a larger uncertainty on this parameter.

32. Our fourth example applies Bayesian inference and model comparison techniques to de inference of coronal density scale height and magnetic field expansion from observations of multiple period transverse oscillations.

33. Some 10 years ago, the existence of multiple mode coronal loop oscillations was discovered. Verwichte et al. detected the presence of both the fundamental and the first harmonic kink mode in a number of loops belonging to a coronal arcade. This detection opened the way to perform a seismological analysis to determine the density scale-height in the corona, using MHD oscillations. In a uniform cavity, the spectrum of oscillations is uniformly distributed and the ratio of periods P1/2P2 should be equal to one. However, because of the presence of density stratification in the corona, this ratio is smaller than one and directly depends on the density scale height in the corona.

34. By considering an exponentially stratified atmosphere projected onto a semicircular coronal loop, one can mimic the stratified atmosphere by projecting the density variation along the loop onto a straight tube model of length L and height at the apex L/pi. Then, one can use the measured period ratio to estimate the coronal density scale height.

35. This is an example of the inversion performed in 2005. The dots represented the inversion curve from which the density scale height can be obtained for a given value of the period ratio.

36. An alternative interpretation for the departure from unity for the period ratio is due to magnetic field expansion of the waveguide, as proposed by Verth and Erdelyi. In this case, expansion produces an increase of the period ratio P1/2P2.

37. In our work, we have considered both models and performed first parameter inference in the Bayesian framework to determine the coronal density scale height and the magnetic expansion factor. We have models 1 and for density stratification and magnetic expansion with an analytical forward problems that relate the period ratio to H and gamma. For parameter inference we compute the posteriors using Gaussian likelihoods and uniform priors and then, as usual, marginalize.

38. These are the results. We see that in both cases well-defined posteriors are obtained. For the coronal density scale height we obtain estimates of 21 and 56 Mm for two considered cases. For magnetic tube expansion factors values of 1.20 and 1.87 are obtained.

39. Then, one has to assess the relevance of such inferences by performing model comparison. It is true that from the mathematical point of view, if the period ratio is > 1 density stratification looks like the plausible mechanism. On the contrary, if the period ratio is >1, one should think that magnetic tube expansion is causing this. However, data are noisy, and the two hypotheses

have to be tested. We have assessed the performance of three different models: M0 for a uniform tube; M1 for density stratification; and M2 for magnetic expansion. The figure on the right shows the marginal likelihoods for the three models. It is clear that M0 is likely for r ~ 1; M1 for r< 1 and M2 for r>1. In order to make a quantitative comparison, we have computed Vayes factors as a function of data. This is done by considering posteriors ratios between the different models and assuming they are all equally likely a priory. Jeffreys scales enables us to give a relevance to the evidence of one model against the other, in terms of evidence that is: no worth more than a bare mention, positive evidence, strong evidence, or very strong evidence.

40. Consider first M1 vs. M0. It is clear from the figure that a period ratio smaller than one is not sufficient evidence for density stratification. Depending on the period ratio and its uncertainty we have different levels of evidence for our model.

41. Something similar happens when comparing model M2 against M0. In this case, we need to go as far a r= 1.16 to have positive evidence for magnetic expansion producing the deviation of period ratio from unity.

42. Finally, we have compared M1 and M2. The figure shows the different levels of evidence for one model against the other as a function of the measured period ratio.

43. Our last example is to show the application of the three levels of Bayesian inference (parameter inference - model comparison - model averaging) to the determination of the cross-field density structure.

44. Analytical expressions for the period and damping of transverse waves can be obtained under the so-called thin tube and thin boundary approximations. The relevant unknown parameters are the internal Alfvén travel time, the density contrast, and the transverse inhomogeneity length-scale. In the expression for the damping time over the period, F is a numerical factor that depends on the radial density profile that has been assumed.

45. We have considered three alternative density models in which the variation of mass density at the non-uniform layer is either sinusoidal, linear, or parabolic. The figures show the cross-field density profile for these models for and fixed value of the density contrast and varying values of the transverse inhomogeneity length-scale. You might think that the exact profile at the layer should have little influence on periods and damping times, but that seems not to be the case, according to a recent study by Soler et al. (2014).

46. These two figures show the result of performing the classic inversion of the three unknown parameters using observed values for period and damping time. This kind of inversion was first performed by Arregui et al. (2007) and leads to a one-dimensional solution curve in the three-dimensional parameter space that connects all the possible values for the unknown parameters that are compatible with the observed data. Soler et al. (2014) have repeated this inversion using the three alternative density models described before. We can see that the classic inversion leads to different solution curves. These curves show the solution to a mathematical problem, that of finding the 1D solution curve. We will see what happens when we find the solution to the inference problem. To this end, we use Bayesian analysis.

47. In parameter inference, we want to infer the unknown physical parameters conditional on the observed oscillation properties. By particularising Bayes theorem to our problem, we have that the full posterior is proportional to the likelihood times the prior. Then, by marginalising the full posterior we obtain the marginal posteriors for each of the three parameters of interest.

48. For parameter inference, we use again the definition of conditional probability and marginal posteriors. Imagine your forward problem is simply that the number c is the product of a and b. We can construct the joint probability of a and b, given c, which is shown in this surface plot. Then, for a fixed value of a, p(b|a,c) is this dashed line here which gives us the probability of b, given a and c. If we wish the probability of b, given c, all we have to do is to integrate for all values of a. For a fixed value of b, p(a|b,c) is this solid line here which gives us the probability of a, given b and c. If we wish the probability of b, given c, all we have to do is to integrate for all values of a. Using this simple method, we can solve our problem.

49. The following curves show the marginal posterior density functions for the three parameters of interest, Alfven travel time-density contrast-transverse inhomogeneity length scale, with the inference performed for each of the three considered density models and by comparing the inversion performed by using analytic and numerical solutions for the forward problem. We see that similar posteriors are obtained regardless of the assumed density profile and that the most important differences arise because of the TTTB approximations.

50. This plots show the same results but now overplotting the posteriors for the three models in each of the left- and right-hand side panels that correspond to TTTB and numerical results. The Alfven transit time inference is the same regardless of the density model used. Something similar happens with the inference for the density contrast, at least under the TTTB approximations. The most significant differences are obtained in the inference of the transverse inhomogeneity length-scale.

51. However, when we summarise the obtained posteriors by means of the median and the errors at the 68% credible region, we find that the differences are negligible and the adopted density model does not influence that much the inference results.

52. Moving to the next level of inference, model comparison enables us to compare the plausibility of the alternative models to explain observed data. This is done by computing posterior ratios on a one-to-one comparison between two models. Assuming that all three models are equally probable a priori, the comparison reduces to the computation of the Bayes factors. Then, a quantitative assessment can be obtained by using Jeffrey's scale that assigns different levels of evidence depending on the Bayes factor.

53. The results from such a comparison are shown in these plots. First, I show an example surface plot of the evidence for the sinusoidal model given the data in the plane with possible values for observed period and damping time. The evidence changes depending on the observed period and damping time. The next three plots show the two-dimensional distribution of Bayes factors for the comparisons between linear vs. sinusoidal, parabolic vs. sinusoidal, and linear vs. parabolic models. The different grey-shaded regions indicate the level of evidence for one model against the alternative. Only for quite strong damping regimes (low values of Td in comparison with P) do we obtain substantial evidence for one model against the alternative. In those regions, the linear model would be the one supported by data.

54. Finally, the third level of Bayesian inference is model averaging. We have seen that the evidence for one model to be preferred over another is not strong enough for many combinations of observed period and damping times. However, the evidence for each of the models is different. Model averaging consist on combining the obtained posteriors to obtain a model-averaged posterior, weighted with the evidence for each model. By taking e.g., M1 as the reference model we can compute the Bayes factors with respect to it and use them in this expression to compute the model averaged posterior for each parameter.

55. Here is a result, for a case with weak damping. The different line styles show the posteriors for the different density models and the symbols the model averaged posteriors.

56. This is another example, for a case with strong damping.

57. In conclusion, we have applied the three levels of Bayesian inference to the problem of obtaining information on the density structuring in coronal waveguides from damped transverse oscillations. Three different models have been considered, sinusoidal, linear, and parabolic, at the non-uniform transitional layer. In spite of the apparent differences from the classic inversion, Bayesian inference led to very similar results. The application of model comparison techniques could enable us to differentiate between the most plausible model in view of observed period and damping time, but only for strongly damped oscillation. Nevertheless, the three models have different levels of evidence and, even if this evidence is not overwhelming, Bayesian model averaging can be applied to obtain a combined posterior for the unknown parameters that includes the individual model evidence.

58. List of references